

MODELS FOR TORSION HOMOTOPY TYPES

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ABSTRACT. Given an integer $n > 1$ and any set P of positive integers, one can assign to each topological space X a homotopy universal map $X^{(P,n)} \rightarrow X$ where $X^{(P,n)}$ is an $(n-1)$ -connected CW-complex whose homotopy groups are P -torsion. We analyze this construction and its properties by means of a suitable closed model category structure on the pointed category of topological spaces.

§0. INTRODUCTION

This article aims to link recent work of Blanc [Bl], Chachólski [Ch], Dror Farjoun [DF96], Hirschhorn [Hir], and Nofech [N93] with parallel advances by Elvira–Hernández [E-H] and Extremiana–Hernández–Rivas [E-H-R]. We exploit a closed model category structure [Q67] on the category Top_* of pointed topological spaces, for each $n \geq 2$ and each set of positive integers P , in which the class of weak equivalences is the class of maps $X \rightarrow Y$ inducing isomorphisms of homotopy groups with mod m coefficients,

$$\pi_r(X; \mathbb{Z}/m) \cong \pi_r(Y; \mathbb{Z}/m), \quad \text{for } r \geq n+1 \text{ and } m \in P.$$

By suitably factoring, in this closed model category, each map of the form $* \rightarrow X$ into a cofibration followed by a trivial fibration, $* \rightarrow X^{(P,n)} \rightarrow X$, one obtains a colocalization functor which we call a (P, n) -CW-approximation. It is indeed reminiscent from the usual CW-approximation, where one associates with any space X a CW-complex K together with a map $K \rightarrow X$ inducing isomorphisms of homotopy groups. The space $X^{(P,n)}$ is built from torsion Moore spaces of type $M(\mathbb{Z}/m, r)$,

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with $r \geq n$ and $m \in P$, by means of a countable sequence of push-outs. Approximations of spaces using Moore spaces as building blocks have also been discussed by Blanc in [Bl], where interesting applications have been given.

The closed model category structure used in our article is directly inspired by the one given in [E-H-R] for the case of ordinary homotopy groups. It does not coincide with the structure studied by Hirschhorn [Hir] and Nofech [N95], [N96], although the associated homotopy categories are indeed equivalent.

Of course, it is also possible to factor each map $X \rightarrow *$ into a cofibration followed by a trivial fibration, $X \rightarrow X_{(P,n)} \rightarrow *$. This yields a localization functor assigning to each X a space whose homotopy groups are uniquely P -divisible in dimensions $r \geq n+1$ and P -torsion-free in dimension n . (An abelian group A is said to be uniquely P -divisible if multiplication by m is an automorphism of A for every $m \in P$, and an element $a \in A$ is said to be P -torsion if there are integers m_1, \dots, m_r in P , not necessarily distinct, such that $m_1 \cdots m_r a = 0$.) Those functors are variants of the classical localization of spaces at sets of primes. We shall not insist in their analysis, as they have been previously discussed by Bousfield [B94], [B96], and Casacuberta–Rodríguez [C-R]. However, we emphasize that the study of such localizations in the framework of abstract homotopy theory is more naturally associated with a different closed model category structure, in which a functorial model for the localization of a space X is obtained by suitably factoring the map $X \rightarrow *$ into a trivial cofibration followed by a fibration. This is precisely the point of view adopted by Quillen in his pioneering work on rational homotopy theory [Q69]; it was exploited further by Bousfield [B75] in connection with homological localization, and by several other authors since then.

This paper intends to be largely self-contained, except for standard input from homotopical algebra. Thus we supply alternative, direct proofs of earlier results due to Blanc [Bl] and Dror Farjoun [DF92], and improve some of them. Notably, Theorem 5.2 below shows that the homotopy groups of $X^{(P,n)}$ coincide with those of the homotopy fibre of the localization map $X \rightarrow X_{(P,n)}$ in all dimensions except possibly in dimension n ; this gives a positive answer to a question raised in [DF92]. Using this fact, we compute $K(A, d)^{(P,n)}$ for any abelian group A and every $d \geq 1$.

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§1. PRELIMINARIES

We shall work in the pointed category Top_* of topological spaces. Thus, all maps will preserve basepoints and $[X, Y]$ will denote the set of pointed homotopy classes of maps from X to Y .

Given any space M , a space X is called M -cellular [DF96], or an M -CW-complex [Bl], if X belongs to the smallest class of spaces which contains M and is closed under pointed homotopy colimits and homotopy equivalences. A map $f: X \rightarrow Y$ is said to be an M -equivalence if the induced map of based mapping spaces

$$map_*(M, X) \rightarrow map_*(M, Y)$$

is a weak homotopy equivalence. As explained in [DF96, 2.B], for every space X there exists an M -equivalence $CW_M(X) \rightarrow X$ where $CW_M(X)$ is an M -CW-complex; see also [Ch]. This map is called an M -CW-approximation to X . On the other hand, a space X is said to be M -null if the space $map_*(M, X)$ is weakly contractible. For every space X there is a homotopy universal map $X \rightarrow P_M X$ into an M -null space; see [B94], [Ch], [DF96, § 1]. This is called an M -nullification of X .

We shall analyze further these concepts in an important special case. For any positive integer m and $n \geq 2$, let $M(\mathbb{Z}/m, n)$ denote the homotopy cofibre of the standard self-map of S^n of degree m , which is an $(n + 1)$ -dimensional CW-complex such that $H_n(M(\mathbb{Z}/m, n)) \cong \mathbb{Z}/m$ and $\tilde{H}_r(M(\mathbb{Z}/m, n)) = 0$ for $r \neq n$. We shall adhere to Neisendorfer's notation [Ne] for homotopy groups with coefficients, by writing

$$\pi_r(X; \mathbb{Z}/m) = [M(\mathbb{Z}/m, r - 1), X],$$

which is a group if $r \geq 3$. It follows that, if $n \geq 2$, then a map $f: X \rightarrow Y$ is an $M(\mathbb{Z}/m, n)$ -equivalence if and only if the induced homomorphisms

$$f_*: \pi_r(X; \mathbb{Z}/m) \rightarrow \pi_r(Y; \mathbb{Z}/m)$$

are isomorphisms for $r \geq n + 1$. A space X is $M(\mathbb{Z}/m, n)$ -null if $\pi_r(X; \mathbb{Z}/m) = 0$ for $r \geq n + 1$; this amounts to saying that multiplication by m is a monomorphism in $\pi_n(X)$ and an automorphism of $\pi_r(X)$ for $r \geq n + 1$, since the following sequence is exact [Ne, § 1]:

$$(1.1) \quad \cdots \rightarrow \pi_r(X) \xrightarrow{m} \pi_r(X) \rightarrow \pi_r(X; \mathbb{Z}/m) \rightarrow \pi_{r-1}(X) \xrightarrow{m} \pi_{r-1}(X) \rightarrow \cdots$$

The machinery developed by Quillen in [Q67] and [Q69] provides a suitable framework to discuss CW-approximations and nullifications, yielding explicit models which are functorial in Top_* . Recall that a closed model category \mathcal{C} is a category

endowed with three distinguished families of maps called cofibrations, fibrations, and weak equivalences, satisfying certain axioms. For details, properties, and further terminology we refer the reader to [Q67] and [Q69]. See also the recent survey by Dwyer and Spalinsky [D-S].

A map which is a weak equivalence and a fibration will be called a trivial fibration, and a map which is a weak equivalence and a cofibration will be called a trivial cofibration. Given a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ i \downarrow & & \downarrow p \\ B & \longrightarrow & Y, \end{array}$$

the map $i: A \rightarrow B$ is said to have the left lifting property (LLP) with respect to $p: X \rightarrow Y$ if a map $B \rightarrow X$ exists making both triangles commute. In this situation, one also says that p has the right lifting property (RLP) with respect to i .

§2. A GENERALIZATION

If X and Y are arbitrary pointed spaces, we denote by $X \rtimes Y$ the half-smash product $X \wedge Y^+$, where Y^+ denotes the union of Y with a disjoint basepoint. Thus $X \rtimes I$ is the ordinary pointed cylinder.

In [E-H-R], the following closed model category structures were considered on the category Top_* of pointed topological spaces, for each $n \geq 1$. A map $f: X \rightarrow Y$ is said to be an n -fibration if f has the RLP with respect to the family of inclusions

$$(D^{n+r} \rtimes \{0\}) \cup (S^{n+r-1} \rtimes I) \rightarrow D^{n+r} \rtimes I, \quad \text{for } r \geq 0;$$

a map f is a weak n -equivalence if the induced homomorphisms $\pi_r(X) \rightarrow \pi_r(Y)$ are isomorphisms for $r \geq n$; f is an n -cofibration if it has the LLP with respect to all trivial n -fibrations. As explained in [E-H-R], the corresponding homotopy category is equivalent to the ordinary homotopy category of $(n-1)$ -connected CW-complexes.

These closed model category structures can be generalized in the following way. Let $M = \Sigma M'$ be any space which is the pointed suspension of a CW-complex M' . Consider the following families of maps in the category Top_* of pointed topological spaces.

Definition 2.1. Let $f: X \rightarrow Y$ be a map. We say that

- (i) f is a weak M -equivalence if the induced homomorphisms

$$f_*: [\Sigma^r M, X] \rightarrow [\Sigma^r M, Y]$$

are isomorphisms for $r \geq 0$;

(ii) f is an M -fibration if it has the RLP with respect to the family of maps

$$(C\Sigma^r M' \rtimes \{0\}) \cup (\Sigma^r M' \rtimes I) \rightarrow C\Sigma^r M' \rtimes I \quad \text{for } r \geq 0,$$

where C denotes the pointed cone functor,

(iii) f is an M -cofibration if it has the LLP with respect to every trivial fibration.

Since each map in (ii) is both a CW-inclusion and a homotopy equivalence, every Serre fibre map is an M -fibration. However, in contrast with [Hir] or [N95], an M -fibration need not be a Serre fibre map (for instance, every map between non-connected spaces with the same basepoint component is an M -fibration). As usual, a space X will be called M -fibrant if the map $X \rightarrow *$ is an M -fibration (hence, all spaces are M -fibrant) and X will be called M -cofibrant if the map $* \rightarrow X$ is an M -cofibration.

Proposition 2.2. *A map $f: X \rightarrow Y$ is a trivial M -fibration if and only if it has the right lifting property with respect to the family \mathcal{C} of inclusions*

$$* \rightarrow M, \quad \Sigma^r M \rightarrow C\Sigma^r M, \quad r \geq 0.$$

Proof. Note that, if a map $f: X \rightarrow Y$ has the RLP with respect to $\Sigma^r M \rightarrow C\Sigma^r M$, then in particular every diagram of the following form (where the upper arrow is the constant map) admits a lifting

$$\begin{array}{ccc} \Sigma^r M & \xrightarrow{*} & X \\ \downarrow & & \downarrow f \\ C\Sigma^r M & \longrightarrow & Y. \end{array}$$

Therefore, f has the RLP with respect to $* \rightarrow \Sigma^{r+1} M$ as well. As a consequence, if a map f has the RLP with respect to the maps in \mathcal{C} , then the induced homomorphisms $[\Sigma^r M, X] \rightarrow [\Sigma^r M, Y]$ are isomorphisms for all r , so that f is a weak M -equivalence. In order to check that f is an M -fibration, we use the fact that by glueing together two copies of $(C\Sigma^r M' \rtimes \{0\}) \cup (\Sigma^r M' \rtimes I)$ one obtains a space which is homeomorphic to $\Sigma^{r+1} M'$, while $C\Sigma^{r+1} M'$ is homeomorphic to the space obtained by glueing together two copies of $C\Sigma^r M' \rtimes I$ in the same way.

Conversely, let $f: X \rightarrow Y$ be a trivial M -fibration. Suppose given a commutative diagram of the form

$$\begin{array}{ccc} \Sigma^r M & \xrightarrow{u} & X \\ i \downarrow & & \downarrow f \\ C\Sigma^r M & \xrightarrow{v} & Y \end{array}$$

with $r \geq 0$. Then we may argue as follows; cf. [E-H, 2.4]. Since f is a weak M -equivalence, there is a map $w: C\Sigma^r M \rightarrow X$ such that $w_i = u$ and $fw \simeq v$. Let $H: C\Sigma^r M \times I \rightarrow Y$ be a homotopy with $H\partial_0 = fw$ and $H\partial_1 = v$, where ∂_0, ∂_1 denote the face maps. Using the fact that f is an M -fibration, we can find a homotopy $F: C\Sigma^r M \times I \rightarrow X$ such that $fF = H$, extending both w and the constant map $(x, t) \mapsto u(x)$ for $x \in \Sigma^r M$ and $t \in I$. Then $w' = F\partial_1$ satisfies $fw' = v$ and $w'_i = u$, as desired. A similar argument shows that f has the RLP with respect to the map $* \rightarrow M$, hence completing the proof. \square

Theorem 2.3. *For every space M which is the suspension of a CW-complex, the category of pointed topological spaces together with the above families of weak M -equivalences, M -fibrations, and M -cofibrations, has the structure of a closed model category. \square*

We denote by Top_*^M this closed model category structure on the category Top_* , and thus by $Ho(Top_*^M)$ the category obtained from Top_*^M by formally inverting the family of weak M -equivalences. For pointed spaces X and Y , the set of morphisms from X to Y in the category $Ho(Top_*^M)$ will be denoted by $[X, Y]^M$.

The routine verification of the Quillen axioms CM1 to CM5 in order to prove Theorem 2.3 proceeds as in [D-S, § 8], [E-H-R, § 2], or [Q67, II.3]; compare with the approaches of Hirschhorn [Hir] and Nofech [N95]. In order to construct the factorizations stated in axiom CM5, we resort to Quillen's "small object argument" (see [Q67, II.3.3] or [D-S, 7.12]), using the maps given in Proposition 2.2 above. Hence, the resulting factorizations are functorial.

Notice that, in the process of constructing such factorizations, it suffices to take the colimit of a countable sequence whenever the space M is compact. Otherwise it will normally require transfinite sequences, as in [B75], [Hir], or [J]. However, if the space M is a (possibly infinite) wedge $\bigvee_{\alpha \in \Lambda} M_\alpha$ where each M_α is compact, then one can still avoid the use of transfinite sequences by replacing the family \mathcal{C} in Proposition 2.2 by the family consisting of $* \rightarrow M_\alpha$ and $\Sigma^r M_\alpha \rightarrow C\Sigma^r M_\alpha$ for $r \geq 0$ and all $\alpha \in \Lambda$; further details are given in the next section.

§3. LOCALIZATION AND COLOCALIZATION

If one considers the M -cofibrant space X^M constructed by factoring a map $* \rightarrow X$ into an M -cofibration followed by a trivial M -fibration,

$$* \rightarrow X^M \rightarrow X,$$

by means of the "small object argument", what one has is a functor $(-)^M: Top_* \rightarrow Top_*$ together with a natural transformation $\varepsilon: (-)^M \rightarrow \text{Id}$. This is in fact a

model for an M -CW-approximation in the sense of [DF96]. On the other hand, by factoring each map $X \rightarrow *$ into an M -cofibration followed by a trivial M -fibration,

$$X \rightarrow X_M \rightarrow *,$$

one obtains a functor $(-)_M: Top_* \rightarrow Top_*$ together with a natural transformation $\eta: \text{Id} \rightarrow (-)_M$, yielding a model for M -nullification. The canonical maps $X^M \rightarrow X$ and $X \rightarrow X_M$ will be called colocalization and localization, respectively. In this section we describe some basic properties of colocalization.

Since M -cofibrations are ordinary cofibrations and Serre fibre maps are M -fibrations, it follows from standard arguments (see e.g. Theorem 9.7 in [D-S]) that for all spaces X and Y there is a natural bijection

$$(3.1) \quad [X, Y]^M \cong [X^M, Y],$$

that is, the functor $(-)^M$ is left adjoint to the “identity” functor from $Ho(Top_*)$ to $Ho(Top_*^M)$.

If we suppose in addition that X is M -cofibrant, then, since all spaces are M -fibrant, the set $[X, Y]^M$ is in one-to-one correspondence with the set of homotopy classes maps from X to Y in Top_*^M ; see [Q67, 1.16]. Now, arguing as in [D-S, 4.15] and [D-S, 9.10], we infer from (3.1) that if X is M -cofibrant and Y is any space then there is a natural bijection $[X, Y]^M \cong [X, Y]$. Since weak M -equivalences are isomorphisms in $Ho(Top_*^M)$, we have the following.

Theorem 3.1. *If $f: Y \rightarrow Z$ is a weak M -equivalence, then f induces a bijection $[X, Y] \cong [X, Z]$ for every M -cofibrant space X . \square*

As an immediate consequence, one obtains a broad generalization of the classical Whitehead theorem; see also [DF96, 2.E].

Theorem 3.2. *If X and Y are M -cofibrant spaces, then a map $f: X \rightarrow Y$ is a homotopy equivalence if and only if it is a weak M -equivalence. \square*

Corollary 3.3. *For every space Y , the colocalization map $Y^M \rightarrow Y$ has the following universal properties:*

- (1) *It is homotopy initial among weak M -equivalences $f: X \rightarrow Y$.*
- (2) *It is homotopy terminal among maps $f: X \rightarrow Y$ where X is M -cofibrant. \square*

Corollary 3.4. *If X is M -cofibrant, then the colocalization map $X^M \rightarrow X$ is a homotopy equivalence. \square*

Corollary 3.5. *The adjoint pair*

$$Ho(Top_*^M) \xrightleftharpoons[\text{Id}]{(-)^M} Ho(Top_*)$$

sets up an equivalence of categories between $Ho(Top_*^M)$ and the full subcategory of $Ho(Top_*)$ whose objects are the M -cofibrant spaces. \square

The M -cofibrant spaces are precisely the retracts of M -CW-complexes, since for every cofibrant X the map $* \rightarrow X$ has the LLP with respect to $X^M \rightarrow X$. A more explicit description of M -cofibrant spaces is given in the next section in the special case where M is a wedge of torsion Moore spaces.

Let F be the homotopy fibre of the localization map $X \rightarrow X_M$. Since $X_M \rightarrow *$ is a weak M -equivalence, the map $F \rightarrow X$ is a weak M -equivalence as well. Hence, $F^M \rightarrow X^M$ is a weak M -equivalence and we infer the following result, which will be used for calculations in Section 5.

Theorem 3.6. *Let X be any space and let F be the homotopy fibre of the localization map $X \rightarrow X_M$. Then $F^M \simeq X^M$. \square*

If the space M is an infinite wedge $\bigvee_{\alpha \in \Lambda} M_\alpha$, where each M_α is compact, but M itself is not compact, then the construction of X^M described above will require the use of transfinite sequences in general. However, we can obtain a model for X^M whose construction stops at the first infinite ordinal by proceeding as follows.

Notice that a map $f: X \rightarrow Y$ is a trivial M -fibration if and only if it has the RLP with respect to the family \mathcal{C}' of inclusions $* \rightarrow M_\alpha$ and $\Sigma^r M_\alpha \rightarrow C\Sigma^r M_\alpha$ with $r \geq 0$ and $\alpha \in \Lambda$; cf. Proposition 2.2. Hence, for each space X , we can construct a suitable model for X^M by means of the “small object argument” using the family \mathcal{C}' instead of the family \mathcal{C} displayed in Proposition 2.2. For convenience, we next recall the details of the process used to decompose a given map $f: A \rightarrow X$ into an M -cofibration followed by a trivial M -fibration.

Firstly, we consider all maps of the form $g: M_\alpha \rightarrow X$, with $\alpha \in \Lambda$, and use them to construct a space $X^0 = A \vee (\bigvee_{g,\alpha} M_\alpha)$ equipped with a map $p^0: X^0 \rightarrow X$ which coincides with f on A and with g on the wedge summand labelled with g , for each g . This map $p^0: X^0 \rightarrow X$ has the RLP with respect to $* \rightarrow M_\alpha$ for all $\alpha \in \Lambda$. Next, we construct inductively a sequence

$$X^0 \xrightarrow{j^1} X^1 \xrightarrow{j^2} X^2 \longrightarrow \dots$$

together with maps $p^r: X^r \rightarrow X$ such that $p^r j^r = p^{r-1}$. Assuming that the map p^{r-1} has been constructed, we take all commutative diagrams D of the form

$$(3.2) \quad \begin{array}{ccc} \Sigma^r M_\alpha & \xrightarrow{u_D} & X^{r-1} \\ \downarrow & & \downarrow p^{r-1} \\ C\Sigma^r M_\alpha & \xrightarrow[v_D]{} & X \end{array}$$

with $r \geq 0$ and $\alpha \in \Lambda$, and define $j^r: X^{r-1} \rightarrow X^r$ by the push-out

$$(3.3) \quad \begin{array}{ccc} \bigvee_D \Sigma^r M_\alpha & \longrightarrow & X^{r-1} \\ \downarrow & & \downarrow j^r \\ \bigvee_D C\Sigma^r M_\alpha & \longrightarrow & X^r. \end{array}$$

The map $p^r: X^r \rightarrow X$ is the sum of p^{r-1} and all the maps v_D in diagram (3.2). Passage to the direct limit yields a trivial M -fibration $p: X^\infty \rightarrow X$ and the desired factorization of f as

$$A \rightarrow X^\infty \rightarrow X,$$

where X^∞ is M -cofibrant. In particular, if we choose A to be a point, then $X^\infty \simeq X^M$, by Theorem 3.2.

This construction can be modified in order to obtain substantially smaller (although possibly non-functorial) models for X^M . For instance, it suffices to pick one representative within each pointed homotopy class of maps at each step of the process. Thus, if $f: A \rightarrow X$ is a map of CW-complexes and we use cellular maps in the construction above, then we obtain a factorization $A \rightarrow \bar{X} \rightarrow X$, where $A \rightarrow \bar{X}$ is an M -cofibration, $\bar{X} \rightarrow X$ is a weak M -equivalence (which need not be an M -fibration) and \bar{X} is a CW-complex.

§4. THE CASE OF TORSION MOORE SPACES

In the rest of the paper we specialize to the case where M is a wedge of certain compact, torsion Moore spaces. Thus let P be any set of positive integers, not necessarily prime, and $n \geq 2$ a fixed integer. Let $M = \bigvee_{m \in P} M(\mathbb{Z}/m, n)$. We shall use the notation $Top_*^{(P,n)}$ for the associated closed model category structure, and refer to the corresponding families of maps as weak (P, n) -equivalences, (P, n) -fibrations, and (P, n) -cofibrations, respectively. Likewise, we denote the localization $(-)_M$ by $(-)_{(P,n)}$ and the colocalization $(-)^M$ by $(-)^{(P,n)}$.

Thus, a map $f: X \rightarrow Y$ is a weak (P, n) -equivalence if and only if the induced homomorphisms $f_*: \pi_r(X; \mathbb{Z}/m) \rightarrow \pi_r(Y; \mathbb{Z}/m)$ are isomorphisms for $r \geq n + 1$

and each $m \in P$. Note that, if $P_1 \subseteq P_2$ and $n_1 \geq n_2$, then every weak (P_2, n_2) -equivalence is a weak (P_1, n_1) -equivalence.

Our first aim is to provide an algebraic characterization of (P, n) -cofibrant spaces. We shall discuss primarily the cases when $P = \{p^k\}$ or $P = \{p, p^2, p^3, \dots\}$, where p is a prime and $k \geq 1$. In fact, Theorem 4.4 and Theorem 4.5 below will demonstrate that this is sufficiently general. Thus, let $M = M(\mathbb{Z}/p^k, n)$ or $M = \bigvee_{i=1}^{\infty} M(\mathbb{Z}/p^i, n)$, where p is a prime, $k \geq 1$, and $n \geq 2$.

Recall from [K-M, 3.10] that every torsion abelian group is the direct sum of its primary components, and every abelian p -group of finite exponent is a direct sum of cyclic groups. For a torsion abelian group G and a prime p , we denote by G_p the p -primary component of G .

Lemma 4.1. *Let $f: X \rightarrow Y$ be a map between 1-connected spaces with torsion homotopy groups. Suppose that $\pi_r(X)_p = 0$ and $\pi_r(Y)_p = 0$ for $r \leq n-1$, where p is a prime. Then f induces isomorphisms $\pi_r(X; \mathbb{Z}/p^k) \cong \pi_r(Y; \mathbb{Z}/p^k)$ for $r \geq n+1$ if and only if the induced maps $\pi_r(X)_p \rightarrow \pi_r(Y)_p$ are isomorphisms for $r \geq n+1$ and $\text{Tor}(\pi_n(X), \mathbb{Z}/p^k) \rightarrow \text{Tor}(\pi_n(Y), \mathbb{Z}/p^k)$ is an isomorphism as well.*

Proof. In order to prove the first implication, let F be the homotopy fibre of f . The homotopy groups of F are torsion and $\pi_r(F)_p = 0$ for $r \leq n-2$. Moreover, the assumption made implies that $\pi_r(F; \mathbb{Z}/p^k) = 0$ if $r \geq n+1$. Hence, $\pi_r(F; \mathbb{Z}/p^k) = 0$ for all r , except perhaps for $r = n$ and $r = n-1$. Now we exploit the exact sequence derived from (1.1),

$$(4.1) \quad 0 \rightarrow \pi_r(F) \otimes \mathbb{Z}/p^k \rightarrow \pi_r(F; \mathbb{Z}/p^k) \rightarrow \text{Tor}(\pi_{r-1}(F), \mathbb{Z}/p^k) \rightarrow 0,$$

together with the fact that the homotopy groups of F are torsion, to infer that $\pi_r(F)_p = 0$ for $r \neq n-1$. Thus, the map f induces isomorphisms $\pi_r(X)_p \cong \pi_r(Y)_p$ for all r , except perhaps for $r = n$, and the homomorphism $f_*: \pi_n(X)_p \rightarrow \pi_n(Y)_p$ is injective. This implies that $\text{Tor}(\pi_n(X), \mathbb{Z}/p^k) \rightarrow \text{Tor}(\pi_n(Y), \mathbb{Z}/p^k)$ is injective as well. In order to prove that the latter is surjective, consider the commutative diagram

$$(4.2) \quad \begin{array}{ccc} \pi_{n+1}(X; \mathbb{Z}/p^k) & \longrightarrow & \text{Tor}(\pi_n(X), \mathbb{Z}/p^k) \\ \cong \downarrow & & \downarrow \\ \pi_{n+1}(Y; \mathbb{Z}/p^k) & \longrightarrow & \text{Tor}(\pi_n(Y), \mathbb{Z}/p^k), \end{array}$$

in which the horizontal maps are epimorphisms, and hence the right-hand map is an epimorphism too. The converse is proved using the exactness and naturality of the sequence (4.1). \square

Theorem 4.2. *Let X be a space, p a prime, and $n \geq 2$.*

- (1) *If $P = \{p^k\}$ with $k \geq 1$, then X has the weak homotopy type of a (P, n) -cofibrant space if and only if X is $(n - 1)$ -connected, $\pi_r(X)$ is p -torsion for all r , and $\pi_n(X)$ is annihilated by p^k .*
- (2) *If $P = \{p, p^2, p^3, \dots\}$, then X has the weak homotopy type of a (P, n) -cofibrant space if and only if X is $(n - 1)$ -connected and $\pi_r(X)$ is p -torsion for all $r \geq n$.*

Proof. In both cases, if X is (P, n) -cofibrant then the colocalization map $X^{(P, n)} \rightarrow X$ is a homotopy equivalence, by Corollary 3.4. In the construction of $X^{(P, n)}$ described at the end of Section 3, we see inductively that X^r is $(n - 1)$ -connected for all r . Hence $X^{(P, n)}$ is $(n - 1)$ -connected too. Since the class of p -torsion abelian groups is a Serre class [S] and it is closed under direct limits, it follows from a Mayer–Vietoris argument that the reduced singular homology groups $H_r(X^{(P, n)})$ are p -torsion for all r , and Serre’s version of the Hurewicz theorem [S] ensures that the homotopy groups $\pi_r(X^{(P, n)})$ are p -torsion for all r as well. Moreover, $H_n(X^{(P, n)})$ is an epimorphic image of $H_n(X^0)$; hence, in case (1) the group $H_n(X^{(P, n)})$ is a \mathbb{Z}/p^k -module and therefore $\pi_n(X^{(P, n)})$ is also a \mathbb{Z}/p^k -module.

In order to prove the converse statements in (1) and (2), we need to show that the hypotheses made imply that the colocalization map $X^{(P, n)} \rightarrow X$ induces isomorphisms $\pi_r(X^{(P, n)}) \cong \pi_r(X)$ for all r . But this follows from Lemma 4.1. \square

Notice that $M(\mathbb{Z}/p^2, n)$ is not (P, n) -cofibrant if $P = \{p\}$.

If $P = \{p\}$, then the homotopy category $Ho(Top_*^{(P, n)})$ is equivalent to the homotopy category of $(n - 1)$ -connected CW-complexes such that $\pi_n(X)$ is a \mathbb{Z}/p -vector space and $\pi_r(X)$ is p -torsion for $r \geq n + 1$. This class of spaces was considered by Bousfield in [B94]. It would be interesting to develop algebraic models for their homotopy category; recent work of Goerss [G] has opened the way into this direction.

We next show that the case where P is any set of positive integers can be reduced to the special cases discussed above. We say that a prime p has finite height in the set P if there is a nonnegative integer h such that p^{h+1} does not divide any number $m \in P$. If this is the case, then the *height* of p in P is the minimum of such integers h ; we shall denote it by $h(p)$. Otherwise, we say that p has infinite height in P . The following result generalizes Theorem 4.2.

Theorem 4.3. *Let $n \geq 2$ and let P be an arbitrary set of positive integers. Then a space X has the weak homotopy type of a (P, n) -cofibrant space if and only if X is $(n - 1)$ -connected, $\pi_r(X)$ is P -torsion for all r , and $\pi_n(X)_p$ is annihilated by $p^{h(p)}$ for each prime p which has finite height $h(p)$ in P . \square*

Theorem 4.4. *For every space X and every set P of positive integers, let Q be the union of the sets $\{p, p^2, p^3, \dots\}$ for each prime p of infinite height in P , and $\{p^{h(p)}\}$ for each prime p of nonzero finite height $h(p)$ in P . Then $X^{(P,n)} \simeq X^{(Q,n)}$ for any $n \geq 2$.*

Proof. By Theorem 4.3, the classes of (P, n) -cofibrant spaces and (Q, n) -cofibrant spaces coincide. Hence, our claim follows from Corollary 3.3. \square

Theorem 4.5. *Let P be any set of positive integers and $n \geq 2$. Suppose that P is the union of a family of sets P_i such that the numbers in P_i are mutually prime with the numbers in P_j whenever $i \neq j$. Then, for each space X , we have*

$$X^{(P,n)} \simeq \bigvee_i X^{(P_i,n)}.$$

Proof. Since every weak (P, n) -equivalence is a weak (P_i, n) -equivalence, there is a map $X^{(P_i,n)} \rightarrow X^{(P,n)}$ for each i . These yield together a map

$$(4.3) \quad \bigvee_i X^{(P_i,n)} \longrightarrow X^{(P,n)}.$$

For each index i , the inclusion of $X^{(P_i,n)}$ into $\bigvee_i X^{(P_i,n)}$ induces an isomorphism in homology with coefficients in P_i . Hence, by [Ne, 3.10], it also induces an isomorphism in homotopy with coefficients in P_i , that is, it is a weak (P_i, n) -equivalence. Therefore, the natural map $\bigvee_i X^{(P_i,n)} \rightarrow X$ is a weak (P_i, n) -equivalence for all i , and hence it is a weak (P, n) -equivalence. It follows that (4.3) is a weak (P, n) -equivalence between (P, n) -cofibrant spaces, and thus it is a homotopy equivalence. \square

We finally address the case where M is a wedge of Moore spaces of various dimensions. Observe that if $M_1 = M(\mathbb{Z}/p^{k_1}, n_1)$ and $M_2 = M(\mathbb{Z}/p^{k_2}, n_2)$ satisfy either $n_1 > n_2$ or $n_1 = n_2$ and $k_1 \leq k_2$, then the classes of weak $(M_1 \vee M_2)$ -equivalences and M_2 -equivalences coincide, which implies that $X^{M_1 \vee M_2} \simeq X^{M_2}$, by Corollary 3.3. In order to generalize this fact, the following notation will be convenient. If k is an integer, then we write $M(p, k, n) = M(\mathbb{Z}/p^k, n)$; otherwise, $M(p, \infty, n) = \bigvee_{i=1}^{\infty} M(\mathbb{Z}/p^i, n)$.

Let X be a space and $W = \bigvee_{n \geq 2} \bigvee_{m \in P_n} M(\mathbb{Z}/m, n)$, where each P_n is a set of positive integers, possibly empty. For each prime p , let $n(p)$ be the smallest value of n such that p divides some number in P_n , or omit p from the indexing if it does

not occur in W . Let $h(p)$ be the height of p in the set $P_{n(p)}$ (here we do not exclude the possibility that $h(p) = \infty$). Let $M = \bigvee_p M(p, h(p), n(p))$. Then

$$(4.4) \quad X^W \simeq X^M \simeq \bigvee_p X^{M(p, h(p), n(p))}.$$

Indeed, the first homotopy equivalence follows from the fact that the classes of weak W -equivalences and weak M -equivalences coincide, and the second equivalence is proved as in Theorem 4.5.

Let P be any set of primes and $M = \bigvee_{p \in P} M(p, k_p, n_p)$, where $n_p \geq 2$ and k_p is either a positive integer or ∞ . Then one shows as in Theorem 4.2 that a space X has the weak homotopy type of an M -cofibrant space if and only if

- (1) X is 1-connected,
- (2) $\pi_r(X)$ is P -torsion for all $r \geq 1$,
- (3) $\pi_r(X)_p = 0$ for $r < n_p$, and
- (4) if k_p is finite, then $\pi_{n_p}(X)_p$ is annihilated by p^{k_p} .

As applications, we prove the following results.

Theorem 4.6. *Let P be any set of primes. Let P_1, \dots, P_r be a finite partition of P into mutually disjoint subsets. Let $M_i = \bigvee_{p \in P_i} M(p, k_p, n_p)$, where $n_p \geq 2$ and k_p is either a positive integer or ∞ . Then, for each space X , the inclusion*

$$(4.5) \quad \bigvee_i X^{M_i} \longrightarrow \prod_i X^{M_i}$$

is a weak homotopy equivalence.

Proof. Each projection $\prod_i X^{M_i} \rightarrow X^{M_i}$ induces isomorphisms on homotopy with coefficients in P_i and hence it is a weak $(P_i, 2)$ -equivalence. Likewise, each inclusion $X^{M_i} \rightarrow \bigvee_i X^{M_i}$ induces isomorphisms on homology with coefficients in P_i , and hence it is also a weak $(P_i, 2)$ -equivalence, by [Ne, 3.10]. Since the composite

$$X^{M_i} \longrightarrow \bigvee_i X^{M_i} \longrightarrow \prod_i X^{M_i} \longrightarrow X^{M_i}$$

is the identity for all i , the arrow (4.5) is a $(P_i, 2)$ -equivalence for all i and hence it is a $(P, 2)$ -equivalence. Finally, observe that the domain of (4.5) is $(P, 2)$ -cofibrant and the codomain has the weak homotopy type of a $(P, 2)$ -cofibrant space. \square

This result remains true for an infinite partition of P into mutually disjoint subsets, provided we take $\prod_i X^{M_i}$ to be the *weak* product of the spaces X^{M_i} ; thus, $\pi_n(\prod_i X^{M_i}) \cong \bigoplus_i \pi_n(X^{M_i})$ for all n . This fact, together with Theorem 4.5,

shows that every n -connected space X (where $n \geq 1$) with torsion homotopy groups decomposes, up to weak homotopy equivalence, as a wedge $\bigvee_p X_p$ or also as a weak product $\prod_p X_p$, where each X_p is an n -connected, p -torsion CW-complex.

Given arbitrary spaces X and Y , the natural map $X^{(P,n)} \times Y^{(P,n)} \rightarrow X \times Y$ is a weak (P, n) -equivalence. Hence, there is a map

$$(4.6) \quad (X \times Y)^{(P,n)} \longrightarrow X^{(P,n)} \times Y^{(P,n)},$$

which is also a weak (P, n) -equivalence. Since the domain of (4.6) is (P, n) -cofibrant and the codomain has the weak homotopy type of a (P, n) -cofibrant space (by Theorem 4.3), the map (4.6) is a weak homotopy equivalence. As above, this result remains true for infinite weak products.

§5. CALCULATING (P, n) -CW-APPROXIMATIONS

Fix a set P of positive integers and an integer $n \geq 2$. Recall from Theorem 3.6 that, for every space X , the colocalization $X^{(P,n)}$ is closely related to the homotopy fibre of the localization map $X \rightarrow X_{(P,n)}$. The space $X_{(P,n)}$ is constructed from X by means of a sequence of push-outs involving $(n-1)$ -connected spaces, in the process of factoring the map $X \rightarrow *$ into a (P, n) -cofibration followed by a trivial (P, n) -fibration. Therefore, we have

$$\pi_r(X) \cong \pi_r(X_{(P,n)}) \quad \text{for } r \leq n-1,$$

and $\pi_r(X_{(P,n)}; \mathbb{Z}/m) = 0$ for $r \geq n+1$ and $m \in P$, since $X_{(P,n)}$ is weakly (P, n) -equivalent to a point. By (1.1), this implies that the homotopy groups $\pi_r(X_{(P,n)})$ are uniquely P -divisible for $r \geq n+1$ and $\pi_n(X_{(P,n)})$ is P -torsion-free. Moreover, if we denote by $\mathbb{Z}[P^{-1}]$ the smallest subring of the rationals containing $1/m$ for all $m \in P$, then

$$(5.1) \quad \pi_r(X_{(P,n)}) \cong \pi_r(X) \otimes \mathbb{Z}[P^{-1}] \quad \text{for } r \geq n+1,$$

while $\pi_n(X_{(P,n)})$ is isomorphic to the quotient of $\pi_n(X)$ by its P -torsion subgroup; cf. [B94, 5.2]. We shall use the fact that the P -torsion subgroup of an abelian group A is isomorphic to $\text{Tor}(A, \mathbb{Z}[P^{-1}]/\mathbb{Z})$, since $\mathbb{Z}[P^{-1}]/\mathbb{Z}$ is a direct sum of groups \mathbb{Z}/p^∞ , where p ranges over all primes dividing the numbers in P .

Theorem 5.1. *The homotopy fibre F of the map $\eta: X \rightarrow X_{(P,n)}$ is weakly equivalent to a (P, n) -cofibrant space if and only if the two following conditions are satisfied for every prime p which has finite height $h(p)$ in P :*

- (1) *The p -torsion subgroup of $\pi_n(X)$ is annihilated by $p^{h(p)}$;*
- (2) *$\pi_{n+1}(X) \otimes \mathbb{Z}/p^\infty = 0$.*

Proof. We infer from the homotopy exact sequence associated to $F \rightarrow X \rightarrow X_{(P,n)}$ that F is always $(n-1)$ -connected and its homotopy groups are P -torsion. Thus, if no prime has finite height in P , then F is weakly equivalent to a (P,n) -cofibrant space by Theorem 4.3. In the general case, it follows from (5.1) that there is a short exact sequence for $r \geq n$,

$$(5.2) \quad 0 \rightarrow \pi_{r+1}(X) \otimes (\mathbb{Z}[P^{-1}]/\mathbb{Z}) \rightarrow \pi_r(F) \rightarrow \text{Tor}(\pi_r(X), \mathbb{Z}[P^{-1}]/\mathbb{Z}) \rightarrow 0,$$

which splits because the kernel is a divisible group. Look at the case $r = n$ and observe that, for any abelian group A , the group $A \otimes \mathbb{Z}/p^\infty$ is p -divisible and hence it cannot be annihilated by any power of p unless it is zero. This proves our claim. \square

Note that

$$(5.3) \quad X^{(P,n)} \rightarrow X \rightarrow X_{(P,n)}$$

is a homotopy fibre sequence if and only if conditions (1) and (2) of Theorem 5.1 are fulfilled for every prime which has finite height in P . Of course, this restriction disappears if all primes dividing the numbers in P have infinite height, e.g. if P is multiplicatively closed. In that case, (5.3) is a homotopy fibre sequence for all spaces X .

The following result answers a question left open in [DF92, 6.4], where it was asked if F and $X^{(P,n)}$ differ at most in one homotopy group.

Theorem 5.2. *Let X be any space and $P = \{p^k\}$, where p is a prime. Let F be the homotopy fibre of the localization map $\eta: X \rightarrow X_{(P,n)}$. Then there is a homotopy fibre sequence*

$$X^{(P,n)} \rightarrow F \rightarrow K(\pi, n)$$

where $\pi = \pi_n(F) / \text{Tor}(\pi_n(F), \mathbb{Z}/p^k)$.

Proof. Since F is $(n-1)$ -connected, we have $H^n(F; \pi) \cong \text{Hom}(\pi_n(F), \pi)$, and hence we may pick a map $g: F \rightarrow K(\pi, n)$ inducing the natural projection $\pi_n(F) \rightarrow \pi$. Let F' be the homotopy fibre of g . Then $\pi_r(F') \cong \pi_r(F)$ for $r \geq n+1$, and $\pi_n(F') \cong \text{Tor}(\pi_n(F), \mathbb{Z}/p^k)$. Therefore, the map $F' \rightarrow F$ is a weak (P,n) -equivalence and F' has the weak homotopy type of a (P,n) -cofibrant space. This shows that F' is weakly equivalent to $X^{(P,n)}$. \square

Now the homotopy groups of $X^{(P,n)}$ can easily be computed in terms of the homotopy groups of X , for any $n \geq 2$ and any set P of positive integers. In the case $P = \{p, p^2, p^3, \dots\}$, the homotopy groups of $X^{(P,n)}$ are isomorphic to those of F , and the latter can be read directly from the split exact sequence (5.2). The

case $P = \{p^k\}$ is covered by Theorem 5.2. Finally, by resorting to Theorem 4.4 and Theorem 4.5, one can compute $X^{(P,n)}$ for other sets P of positive integers.

Example 5.3 Let $P = \{p^k\}$, where p is a prime. Then, for any abelian group A and $d \geq 1$, we have

$$K(A, d)^{(P,n)} \simeq \begin{cases} * & \text{if } d \leq n - 1; \\ K(\text{Tor}(A, \mathbb{Z}/p^k), n) & \text{if } d = n; \\ K(B, n) \times K(T_p A, n + 1) & \text{if } d = n + 1; \\ K(A \otimes \mathbb{Z}/p^\infty, d - 1) \times K(T_p A, d) & \text{if } d \geq n + 2, \end{cases}$$

where $B = \text{Tor}(A \otimes \mathbb{Z}/p^\infty, \mathbb{Z}/p^k) \cong A/(p^k A + T_p A)$ and we denote by $T_p A$ the p -torsion subgroup of A . To check this, consider the homotopy fibre F of $\eta: K(A, d) \rightarrow K(A, d)_{(P,n)}$ and use Theorem 5.2. If $d \geq n + 1$, then

$$K(A, d)_{(P,n)} \simeq K(A \otimes \mathbb{Z}[1/p], d)$$

and F is in fact a product

$$F \simeq K(A \otimes \mathbb{Z}/p^\infty, d - 1) \times K(T_p A, d);$$

cf. [B82, § 4]. If $d = n$, then $F \simeq K(T_p A, n)$.

Example 5.4 Let $P = \{p, p^2, p^3, \dots\}$, where p is a prime. Using similar arguments as in the previous example, for any abelian group A and $d \geq 1$, we have

$$K(A, d)^{(P,n)} \simeq \begin{cases} * & \text{if } d \leq n - 1; \\ K(T_p A, n) & \text{if } d = n; \\ K(A \otimes \mathbb{Z}/p^\infty, d - 1) \times K(T_p A, d) & \text{if } d \geq n + 1. \end{cases}$$

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